

# THE SPECTRAL INDEX OF SIGNED LAPLACIANS AND THEIR STRUCTURAL STABILITY

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**Abstract.** Given a graph Laplacian with positively and negatively weighted edges we are interested in characterizing the set of weights that give a particular spectral index, i.e. give a prescribed number of positive, zero, and negative eigenvalues. One of the main results of this paper is that the set of signed Laplacians that exhibit multiple zero eigenvalues is “small”, and that eigenvalue crossings are nongeneric — specifically, eigenvalues repel each other near zero in a sense that can be made precise. We exhibit an algebraic discriminant that measures the level of repulsion, and show that this discriminant admits a combinatorial interpretation. Conversely, we exhibit a constructive method for finding the sets of Laplacians that exhibit a large degree of degeneracy (many eigenvalues at or near zero) in terms of these discriminants.

**Keywords:** Spectral graph theory, dynamics on networks, graph Laplacian, social networks

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**1. Introduction.** Let  $\Gamma = \{\gamma_{ij}\}$  be a symmetric weighted graph, let  $\mathcal{L}(\Gamma)$  be its graph Laplacian, and let  $n_+(\Gamma)$  be the number of positive eigenvalues of this Laplacian. It is a classical result that if  $\gamma_{ij} > 0$ , then  $n_+(\Gamma) = 0$ . In [1], the authors computed bounds for  $n_+(\Gamma)$  when  $\gamma_{ij}$  are allowed to be both positive and negative. We showed that the sign topology of the graph (i.e., only knowing which edges were positive and which were negative) determines strict upper and lower bounds on  $n_+(\Gamma)$ , and, moreover, that these bounds are saturated: there is some choice of weights that achieves these bounds.

There are graphs for which  $n_+(\Gamma)$  is independent of the magnitude of all of the weights (these graphs were termed “rigid” in [1]); however, for “most” graphs,  $n_+(\Gamma)$  depends on the magnitudes of the weights attached to each edge. A natural question is then, for any particular sign topology, to describe the set of weights which give a prescribed number of positive eigenvalues. In this paper, we study this question in its most general formulation, describing these sets for any particular sign topology, as well as considering questions of genericity.

We also consider applications of this question to designing networks with particular dynamical properties, in particular those that can differentiate multiple signals, and those that are stable but can support multiple solutions on long timescales.

**1.1. Results of paper.** If  $\Gamma = \{\gamma_{ij}\}$  be a weighted symmetric graph, then we define  $\mathcal{L}(\Gamma)$  to be its graph Laplacian, i.e.  $\mathcal{L}(\Gamma)$  is the matrix whose entries are

$$\mathcal{L}(\Gamma)_{ij} = \begin{cases} \gamma_{ij}, & i \neq j, \\ -\sum_{k \neq i} \gamma_{ik}, & i = j. \end{cases} \quad (1.1)$$

We define  $(n_-(\Gamma), n_0(\Gamma), n_+(\Gamma))$  to be the number of negative, zero, and positive eigenvalues of  $\mathcal{L}(\Gamma)$ .

Let  $G = (V, E)$  be a symmetric unweighted graph, where all of the edges are colored black or red. We term this colored structure the “topology” of the graph  $G$ . We also define the two subgraphs  $G_+$  (resp.  $G_-$ ) to be the subgraphs where we consider only black (resp. red) edges. We define  $c(\cdot)$  of any graph to be the number of its connected components.

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Let us now choose the convention that black edges will correspond to positive weights, and red edges to negative weights<sup>1</sup>. It is clear that, given a weighted graph  $\Gamma$ , there is a unique colored graph  $G$  associated with it — we simply forget the magnitude of the weights and retain only their signs. Conversely, for any signed graph  $G$ , there is a natural map from the positive orthant to the set of weighted graphs  $\Gamma$ .

We then show a number of results about the index of all weighted graphs with a given topology. We first state the following results, which are a generalization and slight restatement of the authors' previous results in [1]:

1. If we fix the positive weights of the graph and choose the negative weights in a neighborhood of zero, then generically the Laplacian graph has  $c(G_+) - 1$  positive eigenvalues, a single zero eigenvalue, and the rest positive. In particular, if  $G_+$  is connected, then for negative weights sufficiently small, the Laplacian is negative semidefinite.
2. If we fix the positive weights of the graph and choose the negative weights in a neighborhood of  $\infty$ , then the Laplacian graph has  $n - c(G_-)$  positive eigenvalues, a single zero eigenvalue, and the remainder negative. In particular, if there are any red edges, then we can destabilize the matrix by choosing their weights large enough.

If we consider a one-parameter family of Laplacians with fixed black weights and move the red weights from zero to infinity along some ray, then it follows from the above that  $\tau := n - c(G_+) - c(G_-) + 1$  eigenvalues move through zero from left to right. The next question is: does this one-parameter family have  $\tau$  distinct crossings of individual eigenvalues, or do multiple eigenvalues cross at the same time? (Note here that the Laplacian will always have an eigenvalue fixed at 0, so a simple crossing corresponds to an eigenvalue with multiplicity exactly two.)

To state the answer precisely, we need some notation. Let  $G$  be a fixed graph with  $B$  black edges and  $R$  red edges, and denote the weights on these edges by  $w_+ \in \mathbb{R}^B$  and  $w_- \in \mathbb{R}^R$ . This induces a map from  $\mathbb{R}^B \times \mathbb{R}^R$  to all weighted graphs with a given topology, by setting the positive weights to be the values of  $w_+$  and the negative weights to be  $-w_-$ . If we restrict  $w_+, w_-$  to the positive orthant, then there will be positive weights on the black edges and negative weights on the red edges. The mapping from  $\mathbb{R}^B \times \mathbb{R}^R$  to weighted graphs induces a topology on weighted graphs, and it is in this sense that we use the term generic below.

The one-parameter family of graphs described above corresponds to fixed  $w_+$  and the ray  $tw_-$ ,  $t \in [0, \infty)$  — this gives a one-parameter family of graphs, which gives a one-parameter family of sets of eigenvalues, indexed by  $t$ .

The main results of this paper are as follows. We will only state and prove the results in the case where  $\Gamma$  is connected (if not, the Laplacian is the direct sum of the connected copies, so these results generalize in a straightforward manner).

1. Fix any  $w_+$  for the positive weights. For a generic set of  $w_-$ , all of the eigenvalues have distinct crossings, i.e. there are  $\tau$  distinct values of  $t$  at which the Laplacian has a single eigenvalue crossing zero.
2. There exists a generic set of  $w_+$  such that for any  $w_-$ , all of the eigenvalues have distinct crossings. Moreover, for any fixed  $w_+$ , if we consider the set of all  $w_-$  with  $\|w_-\| = 1$ , then there is a “minimum distance” in  $t$  between successive eigenvalue crossings. This shows the phenomenon of level repulsion for generic Laplacians.

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<sup>1</sup>We are stealing this convention from the accounting industry.

3. To obtain multiple eigenvalues crossing simultaneously, then from the previous two statements, one must be “doubly nongeneric” in a certain sense. We introduce an algebraic expression, related to the discriminant of a polynomial, that determines when this nongeneric situation can occur. This discriminant can be expressed in terms of the homology of the graph, and corresponds to a signed count of certain spanning 2-forests in the graph.
4. Finally, we give sufficient  $\ell^1$  conditions on the vector  $w_-$  that guarantee stability of the Laplacian, and moreover characterize these conditions in a combinatorial manner.

There is an obvious duality in the statements above, since the spectrum of  $-L$  is just the spectrum of  $L$  times  $-1$ , so we could also choose to fix the negative weights  $w_-$  and vary  $w_+$  and obtain obvious analogues of the above results.

**1.2. Applications.** We present two applications motivating these questions:

1] Consider the problem of designing a linear network to differentiate multiple signals, where the network structure is prescribed. More specifically, consider the system

$$\frac{dx}{dt} = Ax + f(t), \quad x \in \mathbb{R}^n, \quad f: \mathbb{R} \rightarrow \mathbb{R}^n \quad (1.2)$$

where  $f(t)$  represents an external signal, the matrix  $A$  represents the network connectivity, and the vector  $x(t)$  represents the network response. In order for the system to be stable we require that the spectrum of the matrix  $A$  lie in the left half-plane. It is clear that eigenvectors corresponding to eigenvalues far from zero are difficult to excite, so the main response from such a system is from the eigenvalues near zero. For such modes the system above acts as an integrator, integrating the projection of the external signal onto the eigenmodes. Being able to recognize multiple signals is the same as saying that we would like to choose  $A$  to have a large dimensional kernel — one dimension for each signal we would like to be able to recognize — or, perhaps, to choose  $A$  so that it has many eigenvalues near zero so that the responses track the signal with a slow decay.

Of course, it is not difficult to design a linear system with spectrum wherever we would like: simply choose the eigenvalues, then any matrix similar to the corresponding diagonal matrix would do. However, notice that in general this gives a dense matrix, and naively it is not clear how one can choose the eigenstructure so that the eventual linear system is compatible with a desired topological structure. However, using the results of this paper, we show how this can be accomplished: if we can design a network with a  $k$ -fold degeneracy at zero, then an open set of perturbations of the weights of such a system will give the desired network.

2] Given a graph  $G = (V, E)$  and symmetric coupling functions  $\varphi_{ij}(\cdot) = \varphi_{ji}(\cdot)$ , define

$$\frac{d}{dt}x_i = F_i(\mathbf{x}) := \omega_i + \sum_{(i,j) \in E} \varphi_{ij}(x_j - x_i). \quad (1.3)$$

One famous case of this model is the Kuramoto oscillator network [2–5], where  $\varphi_{ij}(\cdot) = \gamma_{ij} \sin(\cdot)$ . Assume  $\mathbf{x}^*$  is a fixed point for (1.3), i.e.  $F_i(\mathbf{x}^*) = 0$  for all  $i$ . The stability of this point is determined the index of the Jacobian  $J$ , where

$$J_{ij} = \begin{cases} \varphi'_{ij}(x_j^* - x_i^*), & i \neq j, \\ -\sum_k \varphi'_{ik}(x_k^* - x_i^*), & i = j. \end{cases}$$

In the Kuramoto case, the off-diagonal terms are given by  $\gamma_{ij} \cos(x_j^* - x_i^*)$ . The Jacobian  $J$  is a graph Laplacian of the form (1.1); thus, determining the stability indices for fixed points of (1.3) is related to the problem studied here [6–10]. Of course, identifying those fixed points whose Jacobian is negative semi-definite gives the attracting fixed points for the system, but in fact being able to determine which of these points have one unstable eigenvalue is important to understand metastable transitions for stochastic versions of this system [11].

If the components of  $x^*$  are close enough to each other, then  $\cos(x_j^* - x_i^*) > 0$ , and the Jacobian is negative semidefinite by the classical theory (see Theorem 3.1 of [12]). However, one might ask about the stability of “splay states”, i.e. those stationary points where some of the components are far enough from each other to make the  $\cos(\cdot)$  term negative. For a generic choice of  $\omega_i$ ,  $\varphi'_{ij}(x_i^* - x_j^*)$  is non-zero for all  $(i, j) \in E$ , implying that the graph determining  $J$  and the graph defined by the original interactions in (1.3) have the same underlying topology. Thus we have a fixed network topology, and want to understand the effect of some edge weights being negative. The boundary of the region where the matrix is negative-semidefinite with a one-dimensional kernel is, of course, the set of points where the matrix is negative semi-definite with a higher dimensional kernel [10], and this provides yet another motivation for studying this problem.

**2. Statement of main results.** In this section, we present the main results of this paper, leaving the proofs for later sections. Many of the definitions in this section are identical to those of [1], but we include them here for completeness.

**2.1. Weighted graphs, signed graphs, and the Laplacian.** DEFINITION 2.1.

- A **graph**  $G = (V, E)$  is a set  $V$  of vertices and a set  $E \subseteq V \times V$  of edges.
- A **signed graph** is the triple  $G = (V, E, \sigma)$  where  $(V, E)$  is a standard graph, and with a map  $\sigma: E \rightarrow \{\text{red}, \text{black}\}$ .
- A **weighted graph** is the pair  $\Gamma = (V, \{\gamma_{ij}\})$  where  $\gamma_{ij} \in \mathbb{R}$ . The **edges** of  $\Gamma$  are those  $(i, j)$  with  $\gamma_{ij} \neq 0$ , and we say that  $\gamma_{ij}$  is the **weight** of edge  $i \leftrightarrow j$ .

Any weighted graph corresponds to a signed graph in an obvious manner; we will colloquially call this the “topology” of the weighted graph.

DEFINITION 2.2. Given a weighted graph  $\Gamma$ , the **Laplacian** of  $\Gamma$  is the matrix  $\mathcal{L}(\Gamma)$  with

$$\mathcal{L}_{ij} = \begin{cases} \gamma_{ij}, & i \neq j, \\ -\sum_{k \neq i} \gamma_{ik}, & i = j. \end{cases} \quad (2.1)$$

The **index** of (the Laplacian of)  $\Gamma$  is the triple of integers

$$(n_-(\Gamma), n_0(\Gamma), n_+(\Gamma)) \quad (2.2)$$

giving the number of negative, zero, and positive eigenvalues of  $\mathcal{L}(\Gamma)$ .

NOTATION 2.3. If  $G$  is a signed graph, we will denote by  $G_+$  the subgraph containing only the plus edges, and  $G_-$  the subgraph containing only the minus edges, with a similar convention for  $\Gamma_{\pm}$  when considering weighted graphs. We also denote  $c(G)$  as the number of connected components of a graph, so that  $c(G_+)$  is the number of connected components of a signed graph when we only consider positive edges, etc. We only consider connected graphs in this paper, so that  $c(G) = 1$ , but we allow for  $c(G_+)$  and  $c(G_-)$  to be larger than one. If  $\Gamma$  is a weighted graph, then we can associate

it to a signed graph in the obvious way, and thus all of the notions above make sense as well, i.e.  $\Gamma_+$ ,  $c(\Gamma_+)$ , etc.

**2.2. Crossing polynomial and spectral variety.** DEFINITION 2.4. *If  $T$  is a tree, define  $\pi(T)$  to be the product over the edge weights in the tree*

$$\pi(T) := \prod_{i < j, (i,j) \in E(T)} \gamma_{ij}. \quad (2.3)$$

Let  $\Gamma$  be a weighted graph with  $|V(\Gamma)| = N$ , and  $\mathcal{L}(\Gamma)$  be its graph Laplacian. We know that  $\mathcal{L}(\Gamma)$  has a zero eigenvalue, and therefore  $\det(\mathcal{L}(\Gamma)) = 0$ . Order the  $n$  eigenvalues of  $\mathcal{L}(\Gamma)$  so that  $\lambda_1 = 0$ , then define

$$\mathcal{M}(\Gamma) = \frac{(-1)^{N-1}}{N} \prod_{i=2}^N \lambda_i. \quad (2.4)$$

Thus  $\mathcal{M}(\Gamma)$  is proportional to the linear term in the characteristic polynomial of the Laplacian. Also,  $\mathcal{M}(\Gamma) \neq 0$  iff 0 is a simple eigenvalue of  $\mathcal{L}(\Gamma)$ .

With this notation the Kirchhoff matrix tree theorem can be stated as follows:

LEMMA 2.5 (Weighted Matrix Tree Theorem). *Let  $\Gamma$  be a connected, weighted graph, and  $\mathcal{ST}(\Gamma)$  the set of all spanning trees of  $\Gamma$ . Then*

$$\mathcal{M}(\Gamma) = \sum_{T \in \mathcal{ST}(\Gamma)} \pi(T). \quad (2.5)$$

REMARK 2.6. *This is Theorem VI.29 in the text of Tutte [13]: a proof is provided there. Notice that if all of the edge weights are non-negative, then the sum in (2.5) is a sum of positive terms. This is an alternate proof that the kernel of a graph Laplacian with positive weights is negative semidefinite, and has a simple kernel, when the graph is connected. However, once we allow negative weights, the sum on the right-hand side can have cancellations and will not be sign-definite.*

We will denote the vector  $w = (w_+, w_-) \in (\mathbb{R})^B \times (\mathbb{R})^R$  by

$$(s_1, s_2, \dots, s_B, t_1, t_2, \dots, t_R).$$

The  $s_k$  are the weights of the black edges, and the weights on the red edges are  $-t_k$ .

Every spanning tree will have  $N - 1$  edges. Choose a spanning tree  $T$  and for some  $k$ , it will have  $k$  red edges and  $N - k - 1$  black edges. The contribution to the crossing polynomial is a term of the form

$$\pi(T) = (-1)^k s_{i_1} s_{i_2} \dots s_{i_{N-k-1}} t_1 t_2 \dots t_k. \quad (2.6)$$

If we denote  $\mathcal{ST}_k$  as the set of spanning trees with exactly  $k$  red edges, then

$$\mathcal{M}(\Gamma) = \sum_{T \in \mathcal{ST}(\Gamma)} \pi(T) = \sum_{k=0}^{N-1} \sum_{T \in \mathcal{ST}_k(\Gamma)} (-1)^k s_{i_1} s_{i_2} \dots s_{i_{N-k-1}} t_1 t_2 \dots t_k. \quad (2.7)$$

We see from this that  $\mathcal{M}(\Gamma)$  is a homogeneous polynomial of degree  $N - 1$  with alternating signs.

For the remainder of this paper, we will think of the  $s_k$ 's as *parameters* and the  $t_k$ 's as *variables*. Then  $\mathcal{M}(\Gamma)$  is a polynomial in the variables  $(t_1, \dots, t_R)$ , where the

coefficients are set parametrically by combinations of the  $s_k$ 's. This corresponds to fixing the black edges with specific weights, and varying the red edges.

PROPOSITION 2.7. *The polynomial  $\mathcal{M}(\Gamma)$  has terms of degree  $k$  iff*

$$c(G_+) - 1 \leq k \leq N - c(G_-). \quad (2.8)$$

*Proof.* The proof is the same as the proof of Lemma 2.14 from [1], which we repeat here. Every spanning tree must have at least  $c(G_+) - 1$  red edges to connect the components of  $G_+$  together. Conversely, it must have at least  $c(G_-) - 1$  black edges for the same reason. Since every tree has  $N - 1$  edges, it has at least  $c(G_+) - 1$  and at most  $N - c(G_-)$  red edges. This gives the upper and lower bounds.

To see that the polynomial also contains all of the intermediate terms, we can argue in two ways. One argument is to use contraction-deletion repeatedly (see Remark 2.17 of [1]) to see that the coefficients satisfy log-convexity; this implies that all of the intermediate terms appear in the polynomial as well. A more concrete argument: we can note that for any number  $k$  with  $c(G_+) - 1 \leq k \leq N - c(G_-)$ , there is a subforest of  $\Gamma_-$  with exactly  $k$  edges. Since  $\Gamma_+$  is connected, this means that we can extend this subforest to a spanning tree of  $\Gamma$  using only edges from  $\Gamma_+$ , and thus there is a spanning tree of  $\Gamma$  with exactly  $k$  edges, giving the term of degree  $k$  in the polynomial.  $\square$

PROPOSITION 2.8. *All eigenvalues are nondecreasing as a function of any  $t_k$ .*

*Proof.* The most straightforward proof is to use the results of [1]. Consider the ray in  $\mathbb{R}^R$  given by

$$(t_1, t_2, \dots, t_R) = t(\alpha_1, \dots, \alpha_R),$$

with  $\alpha_i > 0$ . This reduces to the framework of [1], where the crossing polynomial was a function of only one independent variable,  $t$ . Theorem 2.10 of [1] implies that along this ray,  $n_+(\Gamma)$  is nondecreasing, and Lemma 2.18 of [1] implies that it strictly increases by one every time we cross the set  $\mathcal{M}(\Gamma) = 0$ . Inside this set, there is always a double eigenvalue at zero.

A self-contained argument is as follows: First note that for any  $x$ ,

$$x^t \mathcal{L}(\Gamma) x = - \sum_{(i,j) \in E} \gamma_{ij} (x_i - x_j)^2.$$

Clearly, if we increase any  $t_k$ , then this will decrease one of the  $\gamma_{ij}$ , thus increasing this quadratic form. From the Courant minimax theorem, this implies that the eigenvalues of  $\mathcal{L}(\Gamma)$  are nondecreasing as a function of  $t_k$ , since all eigenvalues can be written as the maximum of this quadratic form restricted to some subspace.  $\square$

**2.3. Coefficients of polynomial.** In Definition 2.4, we noted that  $\mathcal{L}(\Gamma)$  has a multiple zero eigenvalue iff  $\mathcal{M}(\Gamma) = 0$ , and from Proposition 2.8 this means that eigenvalue crossings occur on the zero variety  $Z(\Gamma) = \{\mathcal{M}(\Gamma) = 0\}$  in  $\mathbb{R}^R$ , i.e. the connected components of  $Z(\Gamma)^c$  in  $\mathbb{R}^R$  are exactly the sets where  $n_+(\Gamma)$  is constant.

DEFINITION 2.9. *For any spanning tree  $T$  of  $\Gamma$ , we define  $\tilde{\pi}(T)$  as the product of the weights of all of the black edges in  $T$ , i.e.*

$$\tilde{\pi}(T) = \prod_{\substack{i < j, \\ (i,j) \in E(T), \\ \gamma_{ij} > 0}} \gamma_{ij}.$$

DEFINITION 2.10. Let  $\Gamma$  be a weighted graph with  $R$  negative edges, and let  $I, J$  be disjoint subsets of  $[R]$ . We define  $S_I^J$  as the set of all spanning trees of  $\Gamma$  which contain all of the red edges in the index set  $I$ , and none of the red edges in index set  $J$ , i.e.

$$S_I^J(\Gamma) := \{T \mid T \text{ is spanning tree of } \Gamma, e_i \in T \ \forall i \in I, e_j \notin T \ \forall j \in J\},$$

and we define

$$\mathcal{S}_I^J(\Gamma) := \prod_{T \in S_I^J} \tilde{\pi}(T).$$

Finally, as shorthand, we will define

$$S_I = S_I^{I^c}, \quad \mathcal{S}_I = \mathcal{S}_I^{I^c},$$

where the complement  $I^c = [R] \setminus I$ .

DEFINITION 2.11 (Deletion and contraction). Let  $\Gamma = (V, E)$  be a weighted graph, and  $e \in E(\Gamma)$  an edge.

- We denote by  $\Gamma \setminus e$  the graph obtained by removing edge  $e$ , and call this the **deletion** of edge  $e$  from  $\Gamma$ .
- If  $e = (v_1, v_2)$  is an edge with  $v_1 \neq v_2$ , we define  $\Gamma.e$  as follows: identify the two vertices  $v_1$  and  $v_2$  as a single vertex  $v^*$ ; for any vertex  $w$  connected to  $v_1$  or  $v_2$ , we define the new edge weight  $\gamma_{v^*, w} = \gamma_{v_1, w} + \gamma_{v_2, w}$ . We call this the **contraction** of edge  $e$  in  $\Gamma$ .

DEFINITION 2.12. Let  $\Gamma$  be a graph with  $R$  red edges. If  $I, J$  are two disjoint subsets of  $\{1, \dots, R\}$ , then the graph  $\Gamma_I^J$  is the graph obtained by first deleting all of the edges in  $I$ , and then contracting all of the edges in  $J$ . As above, we write  $\Gamma_I$  as shorthand for  $\Gamma_I^{I^c}$ .

PROPOSITION 2.13. Using this notation,

$$\mathcal{S}_I(\Gamma) = \mathcal{M}(\Gamma_I). \tag{2.9}$$

The proof is the standard contraction-deletion theorem, q.v. [14, §13.2].

REMARK 2.14. Let  $\Gamma$  be a graph with  $R$  red edges. For any  $I \subseteq [R]$ , the coefficient of the term  $t^I$ , i.e.  $\prod_{k=1}^{|I|} t_{i_k}$ , is  $\mathcal{S}_I(\Gamma)$ . In particular, this allows us to use the compact notation

$$\mathcal{M}(\Gamma) = \sum_{I \in 2^R} \mathcal{S}_I(\Gamma) t^I. \tag{2.10}$$

If we constrain all positive weights to be one, the quantity  $\mathcal{S}_I(\Gamma)$  has a combinatorial interpretation as the number of spanning trees that contain all red edges in  $I$  and do not contain any red edges not in  $I$ .

REMARK 2.15. We do not use (2.10) explicitly in the results below, but is quite powerful computationally: given  $\Gamma, I$ , to compute  $\mathcal{S}_I(\Gamma)$ , we can compute  $\Gamma_I$ , write down its graph Laplacian  $\mathcal{L}(\Gamma_I)$ , then compute the linear term of its characteristic polynomial. This process can be defined solely in terms of matrices, is easily automatable using a computer algebra system, and is relatively inexpensive.

Finally, we state the main theorem of the paper:

THEOREM 2.16. For generic positive weights, all red rays give single eigenvalue crossings. For all positive weights, generic red rays give single eigenvalue crossings.

We prove these theorems in Sections 3 and 4 below.

**3. Multiple eigenvalue crossings, if  $R = 2$ .** In this section, we concentrate on the case where there are exactly two red edges (see Section 4 for the case of more than two red edges). In this section, we write the vector  $(x_1, x_2)$  as  $(x, y)$ .

Let us assume that  $\Gamma_+$  is connected. Then the main theorem of [1], Theorem 2.10, implies that as the weights on the red edges go to zero,  $n_+(\Gamma) = 0$ , and if they are sufficiently large,  $n_+(\Gamma) = 2$ . Only two eigenvalues cross through zero, so either they cross as a degenerate pair, or they do not. In this section we compute the conditions that determine this.

If  $\Gamma_+$  is not connected, then the crossing polynomial will not have a constant term, but after factoring out the lowest-order terms, it will be in the form (3.1), so that analogous statements about genericity also hold.

**3.1. Discriminant.** Here we connect the discriminant of the 2-variable crossing polynomial to the combinatorics of the graph. Recall from before that we have

$$\mathcal{M}(\Gamma) = A_{11}xy - A_{10}x - A_{01}y + A_{00}, \quad (3.1)$$

where  $A_I = \mathcal{S}_I(\Gamma)$ . Since  $\Gamma_+$  is connected,  $A_{00} > 0$ .

DEFINITION 3.1. *The **discriminant** of the polynomial (3.1) is the quantity*

$$\Delta = A_{11}A_{00} - A_{01}A_{10}.$$

LEMMA 3.2. *If  $\Delta \neq 0$ , the zero-set  $Z(\Gamma)$  defines a hyperbola, with the minimum Euclidean distance  $d$  between the branches given by*

$$d = \frac{\sqrt{2\Delta}}{A_{11}}. \quad (3.2)$$

*In the case  $\Delta = 0$  the set  $Z(\Gamma)$  is a reducible variety and degenerates to the union of the lines  $x = A_{01}/A_{11}$  and  $y = A_{10}/A_{11}$ .*

REMARK 3.3. *This clearly illustrates the level-repulsion phenomenon: while we can make one eigenvalue vanish by varying the strength of one of the edges (say  $x$ ) we typically cannot make two eigenvalues vanish by varying the strength of two edges. The typical situation, when  $\Delta \neq 0$ , is that there are two disconnected branches of the zero-set, each corresponding to a single zero eigenvalue. It is only in the degenerate case where  $\Delta = 0$  that we are able to create a double zero eigenvalue.*

**3.2. Proof of Theorems 2.16 for  $R = 2$ .** It is clear from these formulae that the presence of a potential double root is nongeneric, since the weights of the matrix need to satisfy a polynomial equation. Thus for  $R = 2$ , we have proved Theorem 2.16: as long as  $\Delta \neq 0$ , then all rays give single eigenvalue crossings, and this is clearly generic. Moreover, even if  $\Delta = 0$ , the only way to actually obtain the double eigenvalue crossing is along the ray that passes through the point  $(A_{01}/A_{11}, A_{10}/A_{11})$ .

From this, it is clear that a double eigenvalue is doubly nongeneric: we need the discriminant to be zero, and even then, we need to choose the correct ratio of weights on the red edges to hit the sweet spot. See Figure 3.1.

**3.3. Counting forests.** We have proved the main theorems by showing that the discriminant is the quantity that controls how closely the eigenvalues can cross. We now give a combinatorial interpretation to this discriminant.

DEFINITION 3.4. *For any graph, a **spanning 2-forest** is a spanning subgraph with two connected components — in other words, a spanning tree with one edge re-*



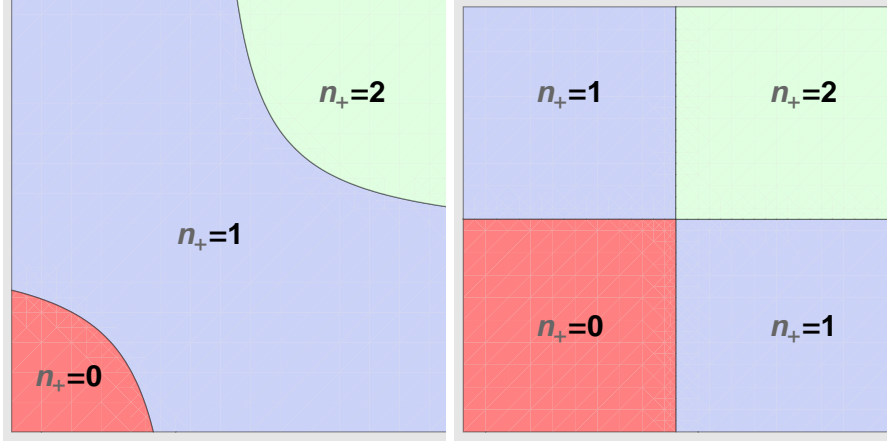


FIG. 3.1. A schematic diagram of the nondegenerate and degenerate cases when there are two red edges.

moved. More generally, a **spanning  $k$ -forest** is a spanning subgraph with  $k$  connected components.

DEFINITION 3.5. Let  $G = (V, E)$  be a signed graph with  $|V| = n$ . Let  $U, W$  be subsets of  $[n]$  with  $|U| = |W| = k$ . We define  $\mathcal{F}_{U,W}^{(k)}$  as the set of all spanning  $k$ -forests such that every tree in the forest has exactly one vertex in  $U$  and one vertex in  $W$ .

We can associate a sign to each forest in  $F \in \mathcal{F}_{U,W}^{(k)}$ , denoted  $\epsilon(F)$ , as follows. Choose and fix some enumeration for  $U, W$ , and define the map  $g: W \rightarrow U$  by saying that  $u = g(w)$  is the unique element of  $U$  that is in the same component of  $F$  as  $w$ . Then  $g(W)$  is a permutation of  $U$  and thus has a sign, and this is  $\epsilon(F)$ .

THEOREM 3.6. Let  $\Gamma$  be a weighted graph with two red edges and where the red edges have weight  $x$  and  $y$ . Let  $i, j$  be the vertices of the first red edge, and  $k, l$  be the vertices of the second. We can form the polynomial (3.1) as above. If we choose  $U = \{i, j\}$  and  $W = \{k, l\}$ , then

$$\Delta = - \left( \sum_{F \in \mathcal{F}_{U,W}^{(2)}} \epsilon(F) \pi(F) \right)^2 \quad (3.3)$$

where  $\pi(F)$  is defined as in (2.3). If all of the black edges have integral weight, then this and (3.2) implies that the Euclidean distance between the two branches of  $Z(\Gamma)$  is  $\sqrt{2}$  times a rational number.

To prove this theorem, we first need some auxiliary results.

THEOREM 3.7 (Chaiken [15]). Let  $G = (V, E)$  be a signed graph with  $|V| = n$ . Let  $U, W$  be subsets of  $[n]$  with  $|U| = |W| = k$ . Define  $|\mathcal{L}_{U,W}|$  to be the minor determinant formed by removing the rows  $U$  and the columns  $W$  from the graph Laplacian  $\mathcal{L}(G)$ . Then

$$|\mathcal{L}_{U,W}| = (-1)^{(\sum_U u + \sum_W w)} \sum_{F \in \mathcal{F}_{U,W}^{(k)}} \sigma(F). \quad (3.4)$$

*Proof.* See Chaiken [15, §3] for a proof in the general (directed graph) case. He uses this result to prove the directed MTT, the directed generalization of (2.5) above.  $\square$

REMARK 3.8. *The case we are particularly interested in here is  $|U| = |W| = 2$ . In this case, forests which assign the least element of  $U$  and the least element of  $W$  to the same component are counted +1, and forests which assign the least element of  $U$  and the least element of  $W$  to different components are counted -1.*

REMARK 3.9. *It is not hard to see that this theorem implies that there are a number of linear relationships among the minor determinants of a graph Laplacian. One example is*

$$|\mathcal{L}_{12,13}| + |\mathcal{L}_{12,14}| = |\mathcal{L}_{12,34}|. \quad (3.5)$$

*To see this note that the first term,  $|\mathcal{L}_{12,13}|$ , is equal to minus the number of 2-forests with vertex 1 in one component and vertices 2 and 3 in the other component. This collection of 2-forests can be split into two types: those in which vertex 4 is in the component with vertex 1 and those in which vertex 4 is in the component with vertices 2 and 3. Similarly the second term,  $|\mathcal{L}_{12,14}|$ , is equal to the number of 2-forests with vertex 1 in one component and vertices 2 and 4 in the other component. This collection of 2-forests can similarly be split into two types: those in which vertex 3 is in the component with vertex 1 and those in which vertex 4 is in the component with vertices 2 and 3. The terms in which vertices 2, 3 and 4 share a component cancel, and we are left with the number of 2-forests in which 1 and 3 share a component and 2 and 4 share a component, minus the number in which 1 and 4 share a component and 2 and 3 share a component. This is exactly  $|\mathcal{L}_{12,34}|$ . A similar identity, which we will use in the proof, is*

$$|\mathcal{L}_{13,13}| + |\mathcal{L}_{14,14}| + |\mathcal{L}_{13,23}| + |\mathcal{L}_{14,23}| = |\mathcal{L}_{12,34}|. \quad (3.6)$$

The next result we use is an identity variously attributed to Dodgson [16], Jacobi [17], and Desnanot [18] on determinants of minor determinants. There is a nice bijective proof due to Zeilberger [19].

THEOREM 3.10 (Dodgson). *Let  $M$  be a matrix, and  $M_{U,W}$  denote the submatrix formed by deleting the rows in  $U$  and the columns in  $W$ . Then*

$$|M| |M_{ij,kl}| - |M_{i,k}| |M_{j,l}| = -|M_{i,l} M_{j,k}|.$$

This identity underlies Dodgson's Method of Condensation, an algorithm for efficient hand computations of determinants. We are now in a position to prove the main theorem:

**Proof of Theorem 3.6.** For this proof, we will use the symbol  $\mathcal{L}_{U,W}$  to be the determinant of the Laplacian matrix with rows  $U$  and columns  $W$  removed.

Let us first assume that the red edges  $x$  and  $y$  do not share a vertex. Renumber the vertices so that edge  $x$  connects vertices 1 and 2 and edge  $y$  connects vertices 3 and 4. First note that any 1-minor determinant is (up to sign)  $P(x, y)$ . In particular we have that

$$P(x, y) = \mathcal{L}_{1,3} = A_{00} + A_{10}x + A_{01}y + A_{11}xy. \quad (3.7)$$

Note that we also have

$$\mathcal{L}_{12,13} + \mathcal{L}_{12,23} = A_{10} + A_{11}y \quad (3.8)$$

and that

$$\mathcal{L}_{13,34} + \mathcal{L}_{14,34} = A_{01} + A_{11}x$$

and finally that

$$\mathcal{L}_{123,134} + \mathcal{L}_{124,134} + \mathcal{L}_{123,234} + \mathcal{L}_{124,234} = A_{11}.$$

Some algebra gives that

$$A_{11}(A_{00} + A_{10}x + A_{01}y + A_{11}xy) - (A_{10} + A_{11}y)(A_{01} + A_{11}x) = A_{11}A_{00} - A_{10}A_{01} = \Delta,$$

or

$$\Delta = (\mathcal{L}_{123,134} + \mathcal{L}_{124,134} + \mathcal{L}_{123,234} + \mathcal{L}_{124,234})\mathcal{L}_{1,3} - (\mathcal{L}_{12,13} + \mathcal{L}_{12,23})(\mathcal{L}_{13,34} + \mathcal{L}_{14,34})$$

and applying the Dodgson identity (here  $\mathcal{L}_{1,3}$  is the matrix  $M$ ) gives

$$\Delta = -\mathcal{L}_{12,34}(\mathcal{L}_{13,13} + \mathcal{L}_{14,13} + \mathcal{L}_{13,23} + \mathcal{L}_{14,23}) = -(\mathcal{L}_{12,34})^2$$

In the case where  $x$  and  $y$  share a vertex we number the shared vertex 1 and the other two 2 and 3. Then a similar calculation to the one above gives

$$\Delta = -(\mathcal{L}_{12,13})^2$$

■

**COROLLARY 3.1.** *Assume  $\Gamma$  has two disjoint red edges,  $x$  and  $y$ , and denote  $x = (i, j)$  and  $y = (k, l)$ . If there is an automorphism of  $\Gamma$  that exchanges  $i, j$  and fixes  $k, l$  (or vice versa), then  $\Delta = 0$ .*

*Proof.* This automorphism fixes  $\pi(F)$  but negates  $\epsilon(F)$ , thus giving  $\Delta = -\Delta$ .  $\square$

**3.4. Cycles and homology.** The expressions derived above give explicit formulae in terms of sums over spanning trees and spanning 2-forests; we now show that there is a dual formula in terms of minor determinants of the cycle intersection form for the homology of the graph  $\Gamma$ .

**THEOREM 3.11.** *Suppose that the original graph  $\Gamma$  has co-rank  $c = E - n + 1$ , and that removing the edges  $x$  and  $y$  does not disconnect  $\Gamma$ . We construct a cycle basis for the graph in the following way: let  $v_1, v_2, \dots, v_{c-2}$  denote a cycle basis for the graph with edges  $x$  and  $y$  removed,  $v_{c-1}$  denote any cycle through  $x$  but not through  $y$  and  $v_c$  denote any cycle through  $y$  but not through  $x$ .*

*In other words the basis should be chosen so that there is a unique cycle through each of the edges  $x$  and  $y$ , but the basis is otherwise arbitrary. Next we define the matrix  $F$  to be the  $E \times c$  matrix with rows given by  $\{v_i\}_{i=1}^c$ . Let  $M_i^j$  denote the minor determinant formed by removing the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of  $FF^t$ . Then  $\Delta = |M_c^{c-1}|^2$ .*

*Proof.* The proof here is analogous to a similar result due to Sjogren [20, Theorem 1] that shows that the determinant of  $FF^t$  is the number of spanning trees of the graph. The proof here is similar in flavor, except that, as we are dealing with a minor determinant, the basis must be chosen in the particular way described above.

Having chosen the basis as above, the main idea is the Cauchy–Binet formula [21, §0.8.7]. We will have to consider various submatrices of  $F$ . Superscripts will denote deleted rows of the matrix (at most one row will be deleted), and subscripts will denote the retained columns. For instance if  $S$  is a subset of the edges then  $F_S^{c-1}$  will

denote the submatrix obtained from  $F$  by deleting the  $(c-1)^{st}$  row (corresponding to the cycle through edge  $x$ ) and deleting all columns not in  $S$ .

Then the Cauchy–Binet formula [21, §0.8.7] gives that  $|M_c^{c-1}| = \sum_S |F_S^{c-1}| |F_S^c|$ , where the sum is over all subsets  $S$  of the edge set  $E$  of size  $c-1$ .

Using Lemma 3.12, the remainder of the proof is straightforward:  $|F_S^{c-1}| |F_S^c|$  unless  $\Gamma/(S \cup y)$  and  $\Gamma/(S \cup x)$  are both trees. This is clearly equivalent to  $\Gamma/(S \cup y \cup x)$  being a spanning 2-forest where  $x$  and  $y$  each have one vertex in each component, so  $|M_c^{c-1}|$  reduces to a signed sum over spanning 2-forests with this property, with the sign given as in the theorem.  $\square$

LEMMA 3.12. *Using the notational conventions established in the statement and proof of Theorem 3.11, we have*

- $|F_S^{c-1}| = 0, \pm 1$ . The determinant is zero unless  $\Gamma/(S \cup y)$  is a tree, in which case  $|F_S^{c-1}| = 0, \pm 1$ , and similarly for  $|F_S^c| = 0$ .
- In the case where  $E/(S \cup y)$  and  $E/(S \cup x)$  are both trees then the sign of  $|F_S^{c-1}| |F_S^c|$  is determined as follows:  $E/S$  is a graph with a unique (up to sign) cycle that contains edges “ $x$ ” and “ $y$ ”. If this cycle traverses edges “ $x$ ” and “ $y$ ” in the same sense then  $|F_S^{c-1}| |F_S^c| = 1$ , otherwise  $|F_S^{c-1}| |F_S^c| = -1$ . Note that this sign is independent of the orientation assigned to edges “ $x$ ” and “ $y$ ” and the direction of the cycle through them.

*Proof.* Let  $c = E - N + 1$  be the co-rank of the graph. We first note the following: if  $\tilde{S}$  is a set of  $c$  edges of  $\Gamma$  then  $|F_{\tilde{S}}| = \pm 1$  if  $\Gamma/\tilde{S}$  forms a tree and  $|F_{\tilde{S}}| = 0$  otherwise. This is proven by Sjogren [20], but for the convenience of the reader we give a quick proof here. First, to see that  $|F_{\tilde{S}}| = 0$  if  $\Gamma/\tilde{S}$  is not a tree we give an explicit element of the kernel. Since  $\Gamma/\tilde{S}$  is not a tree it must contain a cycle. Express this cycle in terms of the basis as  $\sum \alpha_i v_i$ . Thus  $\tilde{\alpha} F_{\tilde{S}}$  gives the edges of this cycle lying in  $\tilde{S}$ . Since this cycle lies entirely in the complement, this is zero.

To see that  $|F_{\tilde{S}}| = \pm 1$  when  $\Gamma/\tilde{S}$  is a tree  $T$  we first consider the case where the basis is chosen in a special way: namely for each edge  $i$  in  $\tilde{S}$  we consider the unique cycle in  $T \cup i$ . This gives a cycle basis. In this particular basis the matrix  $F_{\tilde{S}}$  is a permutation matrix, since each edge in  $\tilde{S}$  lies in a unique cycle. Now *any* integral basis is related to this special basis by a matrix  $U$  with  $|U| = \pm 1$ , and the corresponding matrices are related by  $\tilde{F}_{\tilde{S}} = U F_{\tilde{S}}$ . Therefore for any basis  $|F_{\tilde{S}}| = \pm 1$ .

We are interested in the  $(c-1) \times (c-1)$  matrices  $F_S^c$  and  $F_S^{c-1}$ , where  $S$  is now a subset of  $c-1$  edges, but these matrices can be realized as minors of a matrix  $F_{\tilde{S}}$  of the above form. It is clear that if  $S$  contains the edge  $x$  then  $|F_S^{c-1}| = 0$ , since we have a row of all zeros, and similarly for  $|F_S^c|$ . Since we are only interested in the case where the product is non-zero we can therefore just consider the case where  $S$  does not contain either  $x$  or  $y$ . By the above the matrix  $|F_{S \cup y}| = \pm 1$  if  $\Gamma/(S \cup y)$  is a tree, and zero otherwise. A minor expansion in the row corresponding to  $y$  has a single entry of 1 (in the  $(c, c)$  position), and the rest of the entries are zero. The minor corresponding to the sole nonzero entry is  $F_S^c$ , giving  $|F_S^c| = |F_{S \cup y}| = \pm 1$ . Similarly we have that  $|F_S^{c-1}| = -|F_{S \cup x}| = \pm 1$ , the minus sign arising since the nonzero entry is in the  $(c-1, c)$  position.

To evaluate the relative sign between  $|F_S^c|$  and  $|F_S^{c-1}|$  we give a series of elementary row operations to reduce one to the other. We let  $S$  be an edge subset as above and  $G$  be the  $c \times (c+1)$  matrix consisting of the columns of  $F$  from  $S \cup y \cup x$ . Since  $G$  has a  $c \times c$  submatrix of full rank it has a one dimensional kernel, which can be expressed naturally in terms of the cut-space: Removing the edges  $S \cup x \cup y$  from the graph gives a graph with two components, denoted  $A$  and  $B$ . For each edge in

$S \cup x \cup y$  we associate the following vector  $w$ : if the edge points from  $A$  to  $B$  we assign the corresponding entry of  $w$  to be  $+1$ , if the edge points from  $B$  to  $A$  we assign  $-1$ , otherwise we assign the entry  $0$ . This vector must be in the kernel of  $G$ , since each entry of  $Gw$  gives the number of times that the cycle leaves  $A$  minus the number of times that it leaves  $B$ . This allows us to express the  $c + 1$  column of  $G$  in terms of the first  $c$  columns of  $G$ , and thus gives a sequence of elementary row operations to reduce  $|F_S^{c-1}|$  to  $|F_S^c|$ . The only one which influences the sign of the determinant  $G$  is the orientation of the edges  $x$  and  $y$  relative to the components  $A$  and  $B$ : if the edges  $x$  and  $y$  both point from  $A$  to  $B$ , or vice versa, then  $|F_S^{c-1}| = -|F_S^c|$ , if one edge points from  $A$  to  $B$  and the other  $B$  to  $A$  then  $|F_S^{c-1}| = |F_S^c|$ . Equivalently  $\Gamma/S$  has a cycle containing both edges  $x$  and  $y$ . If this cycle traverses both edges in the same sense the determinants have the same sign, otherwise they have opposite signs.

□

EXAMPLE 3.13. *One graph for which it is straightforward to compute everything is the complete graph,  $K_n$ . We consider the case where the graph has two red edges ( $x$  and  $y$ ) and the rest of the edges have weight 1. In this case it is clear that there are two topologically distinct situations, the case where the two edges share a vertex and the case where they do not. In the case where the two red edges share a vertex it is straightforward though somewhat tedious to check that the polynomial is given by*

$$P_{K_n}(x, y) = 3n^{n-4}xy + (2n-3)n^{n-4}x + (2n-3)n^{n-4}y + (n-1)(n-3)n^{n-4}$$

and the discriminant is then  $\Delta_{K_n} = n^{2n-6}$ .

In the second case, where the edges do not share a vertex, the polynomial is given by

$$P_{K_n}(x, y) = 4n^{n-4}xy + (2n-4)n^{n-4}x + (2n-4)n^{n-4}y + (n-2)^2n^{n-4}$$

and the corresponding discriminant vanishes:  $\Delta_{K_n} = 0$ . (Of course, this vanishing is implied by Corollary 3.1; there are many automorphisms of the complete graph which flip one red edge and leave the other fixed.)

**4. Multiple eigenvalue crossings** —  $R > 2$ . Let us now consider the case where there are  $R$  red edges,  $R > 2$ . As above, the crossing polynomial 2.7 is of the form

$$\mathcal{M}(\Gamma) = \sum_{I \in 2^R} \mathcal{S}_I(\Gamma) x^I.$$

We will find it convenient to index subsets of  $2^R$  by the corresponding binary sequence, and we will write  $A_I = \mathcal{S}_I(\Gamma)$  for brevity.

The variety  $Z(\Gamma) := \{x \in \mathbb{R}^R : M(\Gamma)(x) = 0\}$  can be reducible in many different combinations, but here we concern ourselves only with the maximal notion of reducibility, i.e. we consider the case when  $Z(\Gamma)$  can be written as the union of hyperplanes and the crossing polynomial can be written as a product of linear factors.

It is clear that this is a very non-generic case: a general polynomial  $M(\Gamma)$  is determined by  $2^R$  different coefficients, whereas an  $M(\Gamma)$  that factors is determined by  $R + 1$  different coefficients. This suggests that a necessary and sufficient condition for  $M(\Gamma)$  to factor should be the vanishing of  $2^R - R - 1$  functions of the coefficients. The purpose of this section is to prove this result, and give a nice description of the  $2^R - R - 1$  discriminants whose vanishing is equivalent to factorization of  $M(\Gamma)$ .

DEFINITION 4.1. A **wildcard** sequence is any binary sequence on  $2^R$  with any two of the digits replaced by  $\odot$ 's. Let  $w$  be such a wildcard, and it corresponds with a

set of four binary sequences in  $2^R$  in the canonical way: simply replace the two  $\odot$ 's by the four sequences  $\{00, 01, 10, 11\}$ , call this set  $S_w \subseteq 2^R$ . For any wildcard  $w$ , we define

$$P_w(x) = \sum_{b \in S_w} A_b x^b.$$

We also define  $\Delta_w$ , the discriminant of  $P_w$ , in the obvious manner: it is the product of the coefficients of the highest and lowest order terms, minus the product of the coefficients of the terms of middle order.

EXAMPLE 4.2. If  $R = 5$  and  $w = 00\odot 1\odot$ , then

$$P_w(x) = A_{00010}x_4 + A_{00011}x_4x_5 + A_{00110}x_3x_4 + A_{00111}x_3x_4x_5,$$

and

$$\Delta_w = A_{00010}A_{00111} - A_{00011}A_{00110}.$$

The main result of this section is that the crossing polynomial factors into a product of linear factors if and only if “enough” of the reduced discriminants are zero, and this corresponds to the combinatorial computations of the previous section in a straightforward manner. We first show one direction of this theorem:

PROPOSITION 4.3. If  $\mathcal{M}(\Gamma)$  factors into  $R$  linear factors, then  $\Delta_w = 0$  for any wildcard  $w$ .

*Proof.* We assume that the crossing polynomial factors, so we factor it and then pull out all of the constants in the following manner:

$$\mathcal{M}(\Gamma)(x) = \alpha \prod_{i=1}^R (C_i x_i + 1).$$

We think of  $C = (C_1, \dots, C_R)$  as a vector, and given any vector  $p = (p_1, \dots, p_R)$ , we write

$$C^p = C_1^{p_1} \cdot C_2^{p_2} \dots C_R^{p_R}.$$

From this notation, we see that for any  $b \in 2^R$ ,  $A_b = \alpha C^b$ .

Let  $w$  be a wildcard, and denote the positions of the  $\odot$ 's by  $i < j$ . Let  $b$  be the binary sequence corresponding to  $w$  where we replace both  $\odot$ 's by 0's, and then the four binary sequences correspond to  $w$  are  $\{b, b + e_i, b + e_j, b + e_i + e_j\}$ , where  $e_i$  are the standard basis vectors. Then

$$\Delta_w = C^{b+e_i+e_j}C^b - C^{b+e_i}C^{b+e_j}$$

and this is clearly zero.  $\square$

REMARK 4.4. The previous proposition shows that a complete factorization of  $M(\Gamma)$  implies that  $\binom{R}{2}2^{R-2}$  discriminants vanish. While it is not hard to see that the converse is true, we can prove something stronger. We expect that not all of these discriminants are independent, and the naive count suggests that we need only  $2^R - R - 1$  conditions in order to guarantee that  $M(\Gamma)$  factors. In the next part we define a subset of  $2^R - R - 1$  whose vanishing guarantees that  $M(\Gamma)$  factors.

DEFINITION 4.5. For any  $R$ , we define  $W_R$  (a stacked deck) to be the set of all wildcards of length  $R$  satisfying the following properties:

- The two  $\odot$ 's may be placed in any positions.
- All of the bits before the second  $\odot$  must be zero.
- The bits after the second  $\odot$  are unconstrained: they may be either 0 or 1.

PROPOSITION 4.6. *The set  $W_R$  has  $2^R - R - 1$  elements.*

*Proof.* Let  $w$  be a wildcard where the second  $\odot$  is in slot  $j$ . Then the first  $\odot$  can be chosen in any of  $j - 1$  positions, and all of the entries after  $j$  are free. Thus there are  $2^{R-j}(R - 1)$  wildcards where the second  $\odot$  is in the  $j$ th slot, and therefore

$$|W_R| = \sum_{j=2}^R (j - 1)2^{R-j} = 2^R - R - 1.$$

□

EXAMPLE 4.7. *For  $R = 3$ , we have  $|W_3| = 8 - 3 - 1 = 4$ . The elements of  $W_3$  are*

$$W_3 = \{\odot\odot 1, \odot\odot 0, \odot 0\odot, 0\odot\odot\}. \quad (4.1)$$

*For  $R = 4$ , we have  $|W_4| = 11$ . The elements of  $W_4$  are*

$$W_4 = \{\odot\odot 11, \odot\odot 10, \odot\odot 01, \odot\odot 00, \odot 0\odot 0, \odot 0\odot 1, \odot 00\odot, 0\odot\odot 0, 0\odot\odot 1, 0\odot 0\odot, 00\odot\odot\} \quad (4.2)$$

LEMMA 4.8. *Every sequence in  $W_R$  is of two types:*

- *it is of the form  $w0$  or  $w1$  for some  $w \in W_{R-1}$ ,*
- *it is length  $R$ , it ends with a  $\odot$ , and all non- $\odot$  entries are zeros.*

*Proof.* It is clear that any sequence described above is in  $W_R$ . Then we simply need to count all of the entries described above. There are  $2|W_{R-1}|$  of the first type of sequence, and  $R - 1$  of the second type. Thus we have obtained  $2|W_{R-1}| + R - 1$  sequences of length  $R$ , and this the same recursion relation obtained by the sequence  $|W_R|_{R=3,\dots}$ . □

THEOREM 4.9. *If  $\Gamma$  is a weighted graph with  $R$  negative edges, then a necessary and sufficient condition for  $Z(\Gamma)$  to decompose as a union of hyperplanes is that  $\Delta_w = 0$  for all  $w \in W_R$ .*

*Proof.* The necessary direction of this statement has been proved already in Proposition 4.6, so we consider sufficiency here. We will use a proof by induction, with base case  $R = 2$ ; this case was established in Lemma 3.2.

Assume that the theorem is true for  $R - 1$ , and that  $\Delta_w = 0$  for all  $w \in W_R$ . Let  $b_R \subseteq 2^R$  be all of those sequences with a one in the last slot, and let  $P_R: 2^R \rightarrow 2^{R-1}$  be the projection on sequences that forgets the last slot. Then

$$\mathcal{M}(\Gamma) = \sum_{I \in 2^R} \mathcal{S}_I(\Gamma) x^I = x_R \sum_{I \in b_R} A_I x^{P_R(I)} + \sum_{I \notin b_R} A_I x^{P_R(I)}$$

We write

$$f_R(x) = \sum_{I \in b_R} A_I x^{P_R(I)}, \quad g_R(x) = \sum_{I \notin b_R} A_I x^{P_R(I)}, \quad (4.3)$$

giving

$$\mathcal{M}(\Gamma) = x_R f_R(x) + g_R(x).$$

Note that  $f_R(x), g_R(x)$  are functions only of  $x_1, \dots, x_{R-1}$ .

We can write  $f_R(x)$  as in (4.3), but we can also write this as

$$f_R(x) = \sum_{I \in 2^{R-1}} A_I x^I.$$

Note then that the condition that  $f_R(x)$  fully factor is then that  $\Delta_{w1} = 0$  for all  $w \in W_{R-1}$ , but by Lemma 4.8,  $w1 \in W_R$  for all  $w \in W_{R-1}$ . The argument for  $g_R(x)$  is similar using  $w0$  for  $w \in W_{R-1}$ . Therefore,  $f_R(x)$  and  $g_R(x)$  fully factor into  $R-1$  linear terms.

Now, write  $v, w$  as the vector of coefficients of  $f_R, g_R$ , respectively. Each of these vectors are of length  $2^{R-1}$ . We have shown that each corresponds to a factorizing polynomial, and we needed  $|W_{R-1}| = 2^{R-1} - R$  conditions to establish this. Thus  $v, w$  each only have  $R$  degrees of freedom. Therefore, exactly  $R-1$  additional conditions guarantee that  $v, w$  are linearly dependent. Note that there are  $R-1$  sequences of the second type in Lemma 4.8, and moreover, notice that each of them involve a pair of coefficients that does not appear in any of the others, e.g. the sequence with its first  $\odot$  in the  $i$ th slot involves the coefficients of  $x_i x_R, x_i$ . Thus they are all independent, and these conditions are sufficient to guarantee the collinearity of  $v$  and  $w$ .

We have shown that  $f_R(x), g_R(x)$  each fully factor into  $R-1$  linear terms, and that they are scalar multiples of each other (say  $g_R(x) = \beta f_R(x)$ ). Then we can write

$$\mathcal{M}(\Gamma) = (x_R + \beta) f_R(x),$$

and this is clearly a product of  $R$  linear factors.  $\square$

Finally we note that the arguments of the previous section tell us how to interpret  $\tilde{b}$  as a signed count of spanning forests in a certain derived graph, and this is proved the same way as Theorem 3.6.

**PROPOSITION 4.10.** *Let  $\Gamma_{\tilde{b}}$  be the (multi)graph derived from  $\Gamma$  in the following way: for each 1 digit of  $\tilde{b}$  the corresponding edge of  $\Gamma$  is contracted, while for each 0 digit of  $\tilde{b}$  the corresponding edge of  $\Gamma$  is deleted. Then*

$$\Delta_{\tilde{b}} = \left( \sum_{F \in F^{(2)}} (-1)^{n(F)} \right)^2$$

**5. Stability Estimates.** Here we prove a sufficient condition for stability of a Laplacian in terms of the “worst edge”.

**THEOREM 5.1.** *Let  $\Gamma$  be a weighted graph with fixed positive weights with  $\Gamma_+$  connected and  $R$  red edges. For each red edge  $e_i$ , let  $\Gamma_i = \Gamma_{\{i\}}$  be as defined as in Definition 2.10. Define*

$$\omega_i = \frac{\mathcal{M}(\Gamma_i)}{\mathcal{M}(\Gamma_+)}.$$

*Then, for any  $\mathbf{t} = (t_1, t_2, \dots, t_R)$  with*

$$\|\mathbf{t}\|_{\ell^1} \leq \min_i \omega_i,$$

*define  $G_{\mathbf{t}}$  as the graph where we associate weight  $-t_i$  to edge  $e_i$  has spectral index. Then the spectral index of  $\mathcal{L}(G_{\mathbf{t}})$  is  $(n_-, n_0, n_+) = (N-1, 1, 0)$ .*



*Proof.* For each  $i$ , let  $e_i$  be the standard basis vector with a 1 in slot  $i$ . If  $\mathbf{t} = \alpha e_i$ , then for all  $\alpha < t_i$ ,  $\mathbf{L}(G_{\mathbf{t}})$  is negative semidefinite with  $N - 1$  negative eigenvalues. More generally, if  $\|\mathbf{t}\|_{\ell^1} \leq \min_i \omega_i$ , then  $\mathbf{t}$  is a convex linear combination of the basis vectors described above, and is thus a convex linear combination of positive semidefinite operators, and thus is itself positive semidefinite. Moreover, if we restrict to the subspace  $\mathbf{1}^\perp$ , then the previous sentence is still true after replacing “positive semidefinite” with “positive”.  $\square$

REMARK 5.2. *Basically, this just tells us that if we can rank the red edges from “best” to “worst”, then as long as we would not lose stability by putting all of the weight on the worst edge, then we can redistribute the same amount of negative weights however we choose and still retain stability.*

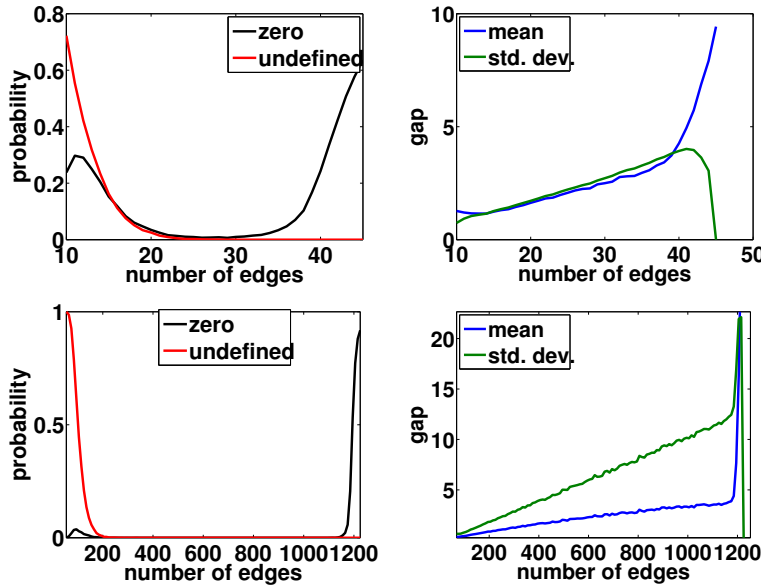


FIG. 6.1. This figure contains several observables for  $G_M(N, M)$  versus  $M$ , where we choose  $N = 10$  in the top row and  $N = 50$  in the bottom. In the left figure, we plot both the probability that  $\Delta = 0$  (black) and the probability that  $\Gamma_+$  is not connected (red). On the right, we plot the mean and standard deviation of the gap conditioned on  $\Gamma_+$  being connected and  $\Delta \neq 0$ .

**6. Numerical Results.** In this section we present a collection of numerical results for random matrices. We are particularly interested in the probability distribution of the nondimensionalized “gap” of a random matrix with two red edges. In all of the simulations performed, we have used the Erdős–Rényi random graphs  $G_M(N, M)$ , the uniform random graph with  $N$  vertices and  $M$  edges. (The specific definition for the distribution on these graphs is as follows. Fix  $N$ , order the  $N(N-1)/2$  edges in some manner, and then choose  $M$  of them without replacement.) To make this a signed graph, we then uniformly choose two of these edges to be red. (We also performed all of the simulations presented here for the graph model  $G_P(N, p)$  (in this case, one chooses each edge to be present independently with probability  $p$ ) and the results are almost indistinguishable. In the interests of space we do not present these here.)

We see a variety of interesting behaviors in these numerics. First, we plot several coarse observables of this ensemble in Figure 6.1. The first quantity of interest is the

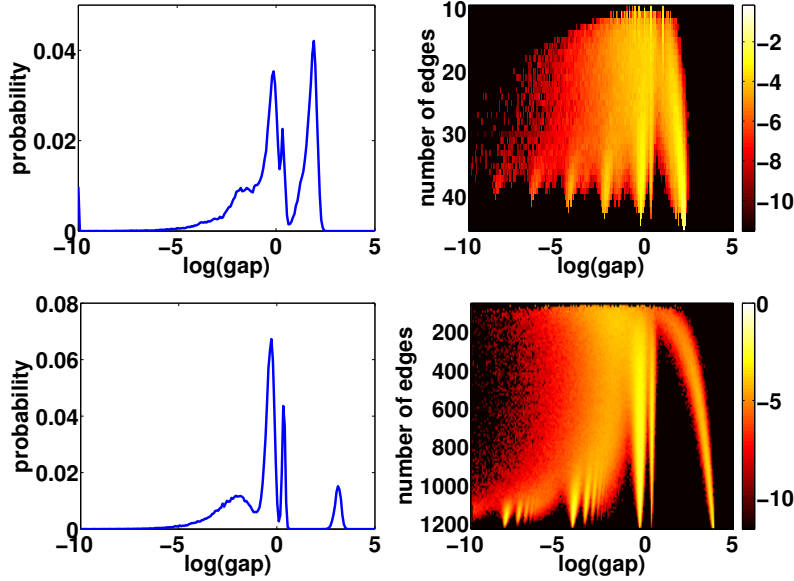


FIG. 6.2. Histograms of the size of the gap for the  $G_M(N, M)$  model.

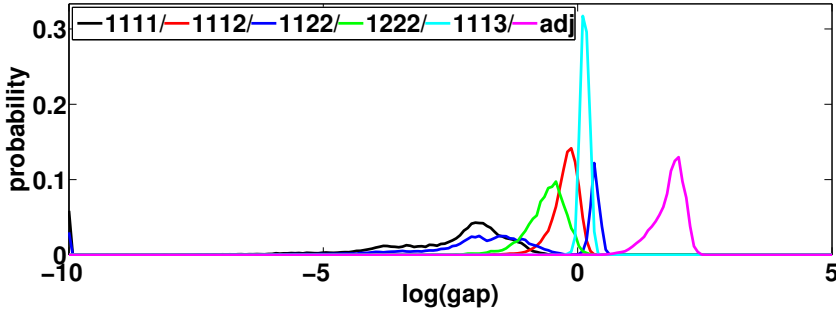


FIG. 6.3. This figure is an overlay of several conditional distributions for  $N = 10, M = 30$ , with specifics described in the text. In particular, the histogram in the top left frame of Figure 6.2 is a convex linear combination of the various histograms presented here.

probability that the gap is zero, and the probability that it is undefined. Recall that the gap will be undefined whenever the graph is disconnected, since in this case  $A_{11}$  is zero (q.v. (3.2)). Of course, this is more likely for a small number of edges, and we see these curves decreasing monotonically. Interestingly enough, the probability of a zero gap is actually not monotone, but in fact turns around: it is most likely when there are very few edges and when there are many edges. In fact, we see that for sparse graphs, the vast majority of the connected graphs have zero gaps. In all of these pictures, the number of edges increases until we reach the complete graph; as proved in the text, the event of having a zero gap for the complete graph is the same as having the two red edges not share a vertex (q.v. Example 3.13). A simple combinatorial argument shows that this probability is  $1 - 4/N$ , giving 0.6, 0.92 in the two graphs plotted in Figure 6.1.

In Figure 6.2, we plot the histograms of the gap in a variety of ways. As in

Figure 6.1, the top row corresponds to  $N = 10$  and the bottom row to  $N = 50$ . In the left column, we have plotted the histogram for a single value of  $M$ ; in the top left, we have  $N = 10, M = 30$  and in the bottom left we have  $N = 50, M = 605$ . In the right column, we plot a heatmap of the histograms for various  $M$ ; in the top right, we plot all  $M$  in the range  $[10, 45]$  and in the bottom right, we have  $M$  going from 55 to 1225 by jumps of 10. Note that in each row, the left frame is a horizontal “slice” of the right frame. Each “slice” in the right figure contains  $10^5$  realizations of random graphs; for the specific histograms in the left column we simulated  $10^7$  random graphs.

What is most striking about this figure is the lack of normality of the distribution, yet, at the same time, the clear impression that they are multimode normal. Moreover, the heatmaps certainly suggest that these modes are coherent as a function of  $M$ . Again, we know that the distribution for the complete graph is degenerate in that it can take on only two values: zero (if the edges are disjoint) and  $N^{2N-6}$  (if the edges share a vertex). Since we are plotting the logarithm of the gap, the zero is represented by a mass at  $-\infty$  (which is here binned at  $-10$ ). So we see that only one nonzero mode “survives” as  $M$  goes to the complete graph, and this is the one corresponding to adjacent edges.

It is natural to ask what these modes correspond to, and we have decomposed the distribution in Figure 6.3. We have done this for the case  $N = 10, M = 30$ , so that this is a decomposition of the histogram in the top left frame of Figure 6.2. What we have done is as follows: for all of the random graphs generated for  $N = 10, M = 30$ , we condition these in various ways. Those where the red edges are adjacent we put in class “adj”. For those in which the red graphs were not, we computed four pairwise path distances between the vertices of the two red edges. To be more specific, if the two red edges were  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$ , we computed  $d_{G_+}(x_i, y_j)$ ,  $i, j = 1, 2$  — that is to say, we computed the graph distance between the nodes using only black edges. If, for example, all of these vertices were adjacent in the black graph, we put it in class “1111”; if, for example, three of them were adjacent, but one pair had a  $G_+$  distance of 2, we put it in class “1112”, etc. We then plotted the conditional distributions for each class, and we see the conditional distributions are close to log-normal (recall that the horizontal axis is always the logarithm of the gap).

In summary, while the distribution of all graphs with a fixed number of edges is multimodal, if we consider those graphs with a fixed number of edges  $M$ , and two red edges, and we condition on the geometry of these red edges, then the conditional distributions are each quite close to log-normal.

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